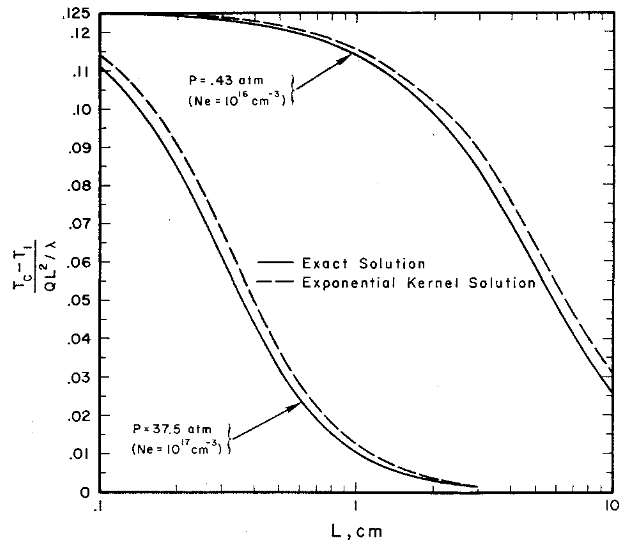

Fig. 2 Centerline temperature results with conduction included, T_1 = 10,000°K.

with the results obtained by making the exponential approximation.[5]

The results for conduction neglected are shown in Fig. 1. In order to see the influence of the exponential approximation, the results for a typical plate spacing can be examined. For a plate spacing of $L = 1$ cm., the exact solution differs from the exponential kernel solution by approximately 20% for T_1 = 10,000°K and P = 0.43 atm; by 15% for T_1 = 10,000°K and P = 37.5 atm; and by 22% for T_1 = 20,000°K and P = 0.57 atm.

Figure 2 shows the results for T_1 = 10,000°K with conduction included. In order to see the influence of the exponential approximation in this case, the results can be examined when radiation and conduction are equally important mechanisms of heat transfer. This occurs when the centerline temperature is reduced to one-half its maximum value, which is the pure conduction value,[5] 0.125. For T_1 = 10,000°K and P = 0.43 atm, the exact dimensionless centerline temperature is reduced by one-half at L = 4.6 cm., and the per cent difference between the exact and exponential kernel solutions is about 10%; for T_1 = 10,000°K and P = 37.5 atm, the difference is 14% at L = 0.29 cm. For T_1 = 20,000°K, the results are similar to the aforementioned results. Thus when conduction is included, the exponential kernel approximation has a relatively small influence on the total heat-transfer problem.

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Integral Solution for Erosion Heat Transfer

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1. Introduction

WHEN a heated solid is in a chemically reactive environment, under excessive pressure and shear forces, or impacted by high-speed particles, erosion may occur at the surface. The erosion phenomena have received substantial attention in recent years due to their importance in high-speed and high-temperature applications. For example, when a vehicle traverses a rain or dust environment, failure may result due to surface erosion coupled with aerodynamic heating. Erosion generally occurs over a wide temperature range and its rate is often strongly dependent on surface temperature. Erosion thus differs from the ordinary ablation process, such as sublimation, which can occur over a range of temperatures but whose rate is determined primarily by the surface heat balance. Moreover, erosion and ablation may occur simultaneously.

In the present analysis, the integral method is applied to solve the transient heat conduction problem involving an eroding surface. The general formulation considers a semi-infinite solid with temperature-dependent properties and an erosion rate that varies arbitrarily with surface temperature and time. Sample closed-form solutions are presented for the cases of erosion with convective heating and of simultaneous erosion and ablation under constant erosive environments.

2. General Formulation

The governing equation for one-dimensional heat conduction in a semi-infinite solid is

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) \tag{1}$$

where the density ρ, specific heat c, and conductivity k are considered as functions of temperature T. The location of the moving surface at time t is defined by $x = S(t)$. With the introduction of

$$z = x - S \tag{2}$$

and

$$\theta = \int_{T_\infty}^{T} \rho c dT \tag{3}$$

Equation (1) becomes

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[\alpha(\theta)\frac{\partial \theta}{\partial z}\right] + \dot{S}\frac{\partial \theta}{\partial z} \tag{4}$$

where $\alpha = k/\rho c$, $\dot{S} \equiv dS/dt$, and the subscript ∞ refers to $x \rightarrow \infty$.

Goodman, who suggested the transformation given by Eq. (3), has applied the integral technique to solve a wide

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variety of problems,[1] including the ablation problem where the surface temperature is a known constant. In the present problem, the surface temperature, heat rate, and surface recession rate are interdependent and are to be determined simultaneously.

In accordance with the integral method, Eq. (4) is integrated from $z = 0$ ($x = S$) to $z = \delta$ where δ is the "heat penetration" distance. Applying the conditions of $\theta = 0$ (i.e., $T = T_\infty$) and $\partial\theta/\partial z = 0$ at $z = \delta$ yields

$$\frac{d}{dt}\int_0^{\delta(t)} \theta dz = -\alpha_0 \left(\frac{\partial\theta}{\partial z}\right)_0 - \dot{S}\theta_0 \tag{5}$$

where the subscript 0 refers to the surface ($z = 0$).

Now a representation for θ must be chosen. The arbitrariness in the selection of a representation is inherent in the nature of approximation with the integral method. Following Goodman we choose a second-degree polynomial in z of the form $\theta = \theta_0 + \theta_1 z + \theta_2 z^2$, where the θ_i may depend on t. Using the conditions of $\theta = 0$ and $\partial\theta/\partial z = 0$ at $z = \delta$, we obtain

$$\theta = \theta_0(1 - z/\delta)^2 \tag{6}$$

where

$$\theta_0 = \int_{T_\infty}^{T_0} \rho c dT \tag{7}$$

Equation (5) then yields

$$(d/dt)(\delta\theta_0/3) = (2\alpha_0/\delta - \dot{S})\theta_0 \tag{8}$$

Relations between δ, θ_0, and $\dot{S}$ are now required. The relations depend on whether or not the surface has reached an ablation temperature T_a. If the surface temperature is below the ablation temperature, then the surface recession is due solely to chemical and/or mechanical erosion. The recession rate can be prescribed as a function of surface temperature and time in the form

$$\dot{S} = \dot{S}(T_0,t) \tag{9}$$

and the surface heat rate can be prescribed as a function of erosion rate, surface temperature, and time in the form

$$Q_0 = Q_0(\dot{S},T_0,t) \tag{10}$$

where $Q_0 = -(k\partial T/\partial x)_0$. Using Eqs. (2, 3, and 6), Eq. (10) becomes

$$2\alpha_0\theta_0/\delta = Q_0 \tag{11}$$

Equations (7–11) can then be used simultaneously to solve for θ_0, T_0, $\dot{S}$, Q_0, and δ as functions of time t. Once θ_0 and δ are known, the temperature distribution can be obtained from Eqs. (3) and (6).

If the surface temperature is sufficiently high so that ablation occurs, e.g., melting, then the surface temperature is known with

$$T_0 = T_a \tag{12}$$

$$\theta_0 = \int_{T_\infty}^{T_a} \rho c dT \tag{13}$$

The surface recession rate now consists of two parts

$$\dot{S} = \dot{S}_e + \dot{S}_a \tag{14}$$

Since the surface temperature is constant, the erosion rate $\dot{S}_e$ can be prescribed as a function of time alone

$$\dot{S}_e = \dot{S}_e(t) \tag{15}$$

while the ablation rate $\dot{S}_a$ must be determined from a heat balance of the form

$$\rho_0 L_a \dot{S}_a = Q_0 - (-k\partial T/\partial x)_0 \tag{16}$$

where L_a is the heat absorbed at the surface per unit mass ablated, and Q_0 is a heat flux that is independent of $\dot{S}_a$ given in the form

$$Q_0 = Q_0(\dot{S}_e,t) \tag{17}$$

Using Eqs. (2, 3, and 6), Eq. (16) becomes

$$\rho_0 L_a \dot{S}_a = Q_0 - 2\alpha_0\theta_0/\delta \tag{18}$$

Equations (8, 14, 15, 17, and 18) determine δ, $\dot{S}$, $\dot{S}_e$, Q_0, and $\dot{S}_a$ simultaneously when the surface is at an ablation temperature.

As an illustration of applying the formulated equations, two closed-form solutions are presented in the following two sections.

3. Erosion with Variable Surface Temperature

For variable surface temperature, Eqs. (7–11) are applicable. Consider the case in which Q_0 and $\dot{S}$ are explicitly independent of time (constant external conditions). Then Eq. (8) can be formally integrated to yield

$$t = \int_0^{\theta_0} \frac{2}{3(Q_0 - \dot{S}\theta_0)} d(\alpha_0\theta_0^2/Q_0) \tag{19}$$

The integration of Eq. (19) can always be carried out since Q_0, $\dot{S}$, and α_0 are known functions of θ_0.

For example, consider the case of constant properties and

$$Q_0 = h(T_f - T_0) - \rho L_e \dot{S} \tag{20}$$

$$\dot{S} = \dot{S}_1 + \dot{S}'_2(T_0 - T_\infty) \tag{21}$$

Equation (20) corresponds to convective heating where h is a heat-transfer coefficient, T_f is the fluid temperature, and L_e is a "heat of erosion" which is defined here as the heat absorbed per unit mass eroded and may have zero value for some mechanical removal processes. Equation (21), in which $\dot{S}_1$ and $\dot{S}'_2$ are constants, represents an erosion rate that varies linearly with surface temperature. Equation (19) now becomes

$$t = \frac{4\rho c k}{3a^2}\int_0^{\gamma} \frac{(1 - b\gamma/2)\gamma d\gamma}{(1 - b\gamma)^2(1 - \bar{c}\gamma - d\gamma^2)} \tag{22}$$

where

$$\gamma = (T_0 - T_\infty)/(T_f - T_\infty) \tag{23}$$

and

$$\begin{aligned} a &= h - (\rho L_e \dot{S}_1)/(T_f - T_\infty) \\ b &= (h + \rho L_e \dot{S}'_2)/a \\ \bar{c} &= (h + \rho L_e \dot{S}'_2 + \rho c \dot{S}_1)/a \\ d &= \rho c \dot{S}'_2(T_f - T_\infty)/a \end{aligned} \tag{24}$$

Equation (22) yields the solution of

$$t = \frac{\rho c k}{3a^2 A}\left[\frac{2b\gamma}{1 - b\gamma} - B\ln\tau_1 + \frac{B^2 - 2C - 2}{(B^2 - 4C)^{1/2}}\ln\tau_2\right] \tag{25}$$

where

$$A = b^2 - b\bar{c} - d,\ B = (b\bar{c} + 2d)/A,\ C = -d/A \tag{26}$$

and

$$\tau_1 = [(1 - b\gamma)^{-2} + B(1 - b\gamma)^{-1} + C]/(1 + B + C)$$

$$\tau_2 = \frac{\left[\frac{2}{1 - b\gamma} + B - (B^2 - 4C)^{1/2}\right][2 + B + (B^2 - 4C)^{1/2}]}{\left[\frac{2}{1 - b\gamma} + B + (B^2 - 4C)^{1/2}\right][2 + B - (B^2 - 4C)^{1/2}]} \tag{27}$$

Equation (25) relates t to T_0 through $t = t(\gamma)$ and Eq. (23). The solutions for $\dot{S}$ and δ are

$$\dot{S} = \dot{S}_1 + \dot{S}'_2(T_f - T_\infty)\gamma \tag{28}$$

$$\delta = \frac{2k(T_f - T_\infty)\gamma}{h(T_f - T_\infty)(1 - \gamma) - \rho L_e\dot{S}} \tag{29}$$

For the case of $\dot{S}'_2 = 0$, the original governing equations become linear and the exact solution is known.[2] It is found that the present solution agrees very well with the exact solution.

4. Erosion with Ablation

Once the surface has reached an ablation temperature, Eqs. (8, 14, 15, 17, and 18) are applicable. Since the surface temperature is now known, Q_0 and $\dot{S}_e$ are functions of time only. Consider again the case of constant external conditions, so that Q_0 and $\dot{S}_e$ are constant. Then Eq. (8) can be integrated to yield

$$t - t_a = \int_{\delta a}^{\delta} \frac{\delta}{3}\, d\delta \Big/ \left[2\alpha_0\left(1 + \frac{\theta_0}{\rho_0 L_a}\right) - \left(\dot{S}_e + \frac{Q_0}{\rho_0 L_a}\right)\delta\right] \tag{30}$$

where t_a is the time when ablation starts and δ_a is the value of δ at $t = t_a$. For simplicity, consider constant properties and let $Q_0 = F - \rho L_e \dot{S}_e$. Equation (30) yields

$$t - t_a = \frac{N(\delta_a - \delta) - \ln[(1 - N\delta)/(1 - N\delta_a)]}{6\alpha[1 + c(T_a - T_\infty)/L_a]N^2} \tag{31}$$

and Eqs. (14) and (18) yield

$$S = S_a + \frac{\delta_a - \delta}{3} - \frac{\ln[(1 - N\delta)/(1 - N\delta_a)]}{3N[1 + c(T_a - T_\infty)/L_a]} \tag{32}$$

where S_a is the value of S at $t = t_a$ and

$$N = \frac{\dot{S}_e(1 - L_e/L_a) + F/\rho L_a}{2\alpha[1 + c(T_a - T_\infty)/L_a]} \tag{33}$$

It may be noted here that as $t \to \infty$, δ and $\dot{S}$ approach the steady-state values of $\delta \to 1/N$ and $\dot{S} \to 2\alpha N$.

It can easily be shown that when $\dot{S}_e = 0$, Eqs. (31) and (32) reduce to the solution for phase change alone as obtained by Goodman.[1] Examination of the results shows that for $\dot{S}_e \neq 0$, the present solution can be equated to Goodman's solution by letting F of his solution be $F + \rho(L_a - L_e)\dot{S}_e$ for the present solution, and by adjusting his integration constant to account for the difference in δ_a. Thus, in the special case when the heat of erosion is equal to the heat of ablation, the two solutions have identical form. It has been stated by Goodman that his results are impossible to distinguish, on a plotted scale, from the exact numerical solution of Landau.[3] One may infer from this statement that the present results are also quite accurate.

5. Discussion

The formulation presented in Sec. 2 allows one to determine the transient temperature distribution, erosion rate, and ablation rate under general erosive environments and arbitrary heating conditions. The method of solution consists of solving a first-order ordinary nonlinear differential equation along with several nonlinear algebraic equations. The numerical computation can be carried out quite simply. The closed-form solutions presented in Secs. 3 and 4 illustrate the general features of heat conduction with an eroding surface.

It is noted that while condensed phase changes usually occur at a unique temperature, erosion may occur over a wide range of surface temperatures. Also, the rate of erosion is in general a prescribed function of surface temperature and time, but the rate of phase change is determined by a heat balance at the surface. Furthermore, erosion and phase change may occur simultaneously and can be treated as shown.

For erosion problems involving pulselike environments, finite slabs, etc., the special techniques illustrated by Goodman[1] may be employed. For erosion rates that depend on surface heat flux and "char thickness" as well as surface temperature and time, the integral approach is an ideal one since both heat flux and char thickness can in turn be related to the surface temperature and heat penetration distance. Finally, it may be noted that if a different representation, such as a higher order polynomial or an exponential profile, is used for the temperature distribution, the governing differential equation will retain the same form with modified coefficients.

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Stability of Linear Regulators Optimal for Time-Multiplied Performance Indices

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I. Introduction

It is well-known that the optimal control law for the time-invariant linear feedback system,

$$\dot{x} = Fx + Gu, \quad x(0) = x_0 \tag{1}$$

which minimizes the quadratic performance index

$$J = \int_0^\infty (x^T Q x + u^T u)dt \tag{2}$$

is given by

$$u^* = -K^T x = -(PG)^T x \tag{3}$$

and P is defined as the solution of

$$-\dot{P} = PF + F^T P - PGG^T P + Q \tag{4}$$

with the initial condition

$$\lim_{T \to \infty} P(T) = 0 \tag{5}$$

Anderson and Moore[1] have shown under assumptions of complete controllability and complete observability that a large amount of nonlinearity may be tolerated in the optimal control law without causing the system to become unstable. In Ref. 2, this result is extended to systems optimal with respect

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